

UPLIFT, FRICTION AND CAPILLARITY: THREE FUNDAMENTAL EFFECTS FOR LIQUID-SATURATED POROUS SOLIDS

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Abstract—In the present paper, the effects of uplift, friction and capillarity for liquid-saturated porous solids are discussed by use of general porous media theories (mixture theories extended by the volume fraction concept). Preceded by several historical remarks on former approaches to a.m. effects, the present investigation is based on a macroscopic binary model of incompressible constituents where, in the constitutive range, use is made of the second-grade character of general heterogeneous media. As a direct consequence of this procedure, the magnitudes of uplift, friction and capillarity effects are easily determined, thus solving an old problem in applied engineering.

1. INTRODUCTION

Motivated by a public discussion about the state of “heavy-weight” masonry dams (“Der Spiegel”, 1986), the present article concerns the discussion of three fundamental mechanical effects for liquid-saturated porous solids, namely the effects of uplift, friction and capillarity by applying general porous media theories to practical civil engineering problems.

In particular, the problem of uplift seems to be very important, since the “heavy-weight” masonry dams of Prof. Intze† and his disciples built about the turn of the century, in a certain sense, run the risk of breaking, compare, e.g. the well-known accidents of Epinal, France in 1895 or Austin, Pennsylvania, U.S.A. in 1911 (Vogel, 1982). However, several dams built using the Intze principle are still functioning all over the world. Although dams as well as other liquid-saturated porous solids are subjected to the “principle of Archimedes”, Intze *et al.* neglected the effect of uplift. Thus, in their calculations, they used the total weight of the dam and not the weight reduced by uplift. This crucial fact led to the result that later, several scientists elaborated on the uplift problem; at first especially in connection with dams and in the following period with respect to general porous media. Concerning the forces acting upon the solid phase of liquid-saturated porous media, the main question that had to be answered was: “Is it necessary to take into account the full value of the uplift force; is it sufficient to use any reduced value or is it even admissible to neglect the uplift completely?” In applied engineering, this question arises not only in connection with masonry or concrete dams but also for soil mechanical problems, e.g. in connection with seepage flow through earth dams.

In investigating the uplift question, it was only natural that other mechanical problems occurred to be discussed, namely the questions of friction and capillarity. There has never been any doubt about the existence of all these effects for liquid-saturated porous media, but the magnitude of uplift, friction and capillary forces was in dispute. Concerning the question of uplift one came to realize that, with respect to the principle of Archimedes, the unreduced value must be effective. On the other hand, the different proofs of this result are difficult to understand, since some of them are based, e.g., on certain investigations of strength of materials, whereas others use special intersecting techniques to arrive at the desired result.

Nowadays, however, the question of the magnitude of uplift, friction and capillary forces can be answered on the basis of commonly accepted thermodynamical principles by use of porous media theories (mixture theories extended by the volume fraction concept).

† Otto Intze (1843–1904) was Professor at the RWTH Aachen and in his time one of the famous engineers for dam constructions.

Concerning a broad review of mixture theories (heterogeneously composed continua with internal interactions) and porous media theories, respectively, the reader is referred to e.g. Bowen (1976), Atkin and Craine (1976a,b), Drumheller (1978), Bowen (1980, 1982), Bedford and Drumheller (1983), Bowen (1984), de Boer and Ehlers (1986, 1988a,b) or Ehlers (1988, 1989b,c).

Within the framework of porous media theories, a macroscopic formulation of liquid-saturated porous solids is employed by the assumption of a binary mixture of superimposed but immiscible constituents. In assuming a statistical distribution of the single constituents through the control space, all mechanical quantities of the model are described via average functions of the macroscale. Since the solid material itself can be considered to be incompressible (the constituent compressibility of the matrix material, in general, is much smaller than the compressibility of the average bulk material), the theory is based on an incompressible binary model governed by a single temperature.

In the scope of general porous media theories, the development of thermodynamical restrictions and constitutive equations must be based on the assumption of second-grade materials, thus avoiding "simple" results as concluded from former approaches to porous media, compare, e.g. the different publications of Biot cited in de Boer and Ehlers (1988a). In the literature, the concept of second-grade materials was introduced to mixture theories by Müller (1968) and discussed in more detail by Cross (1973). Concerning incompressible porous media models, materials of second grade have been used by Kenyon (1976), Bowen (1980, 1984) and Ehlers (1988, 1989b,c).

Proceeding from these approaches, the contribution of the present article is to show how the effects of uplift, friction and capillarity apply to the general theory stated above. To obtain a survey of former discussion on uplift, friction and capillarity effects in porous media (including the different errors that have been made), Section 2 offers some historical remarks, however, without asserting the claim of completeness. With respect to porous media theories, Section 3 concerns a brief review of the fundamental and constitutive equations of an incompressible solid matrix saturated by an incompressible viscous pore liquid. From these results, the question of the magnitude of uplift, friction and capillary forces is clarified in Section 4, thus solving an old problem in applied engineering.

2. HISTORICAL REMARKS

The scientific discussion on uplift in porous media started when Fillunger (1913) specified the uplift force especially with respect to dam constructions, viz. :

$$P_1 = kV(n^F - n^{F'}). \quad (1)$$

In this equation, k defines the gradient of the fluid pressure which Fillunger assumed to be constant and V is the total volume of the liquid-saturated porous solid. The quantities n^F and $n^{F'}$ are the volume and surface porosity coefficients (Fillunger, 1913, used the symbols μ and μ' instead of n^F and $n^{F'}$). In Fillunger (1913) and several subsequent publications (Fillunger, 1914, 1929, 1930, 1935), the different possibilities of (1) concerning the value of the uplift force were discussed. Following this, the difference $n^F - n^{F'}$ can be positive, zero or negative only depending on different intersecting techniques, see Fig. 1: if the porous medium is intersected in an arbitrary manner, i.e. statistically (statistical cut), the Delesseian law, cf. Delesse (1848) and Fillunger (1935), yields $n^F = n^{F'}$ and thus, the uplift force vanishes. On the other hand, if a granular porous medium with point contacts between the grains is intersected exactly through these contact points only (grain-to-grain cut), then $n^{F'} = 1$ and thus, the total uplift force is effective. Finally, if $n^{F'} = 0$ which implies a porous medium with a closed solid surface, the "uplift force" yields in the same way as a load for the medium under discussion. Following Fillunger, $0 \leq n^{F'} \leq 1$ contains all possibilities which may occur in practice. Analogous with (1), Fillunger's uplift force with respect to a volume element of the porous medium is (Fillunger, 1914):

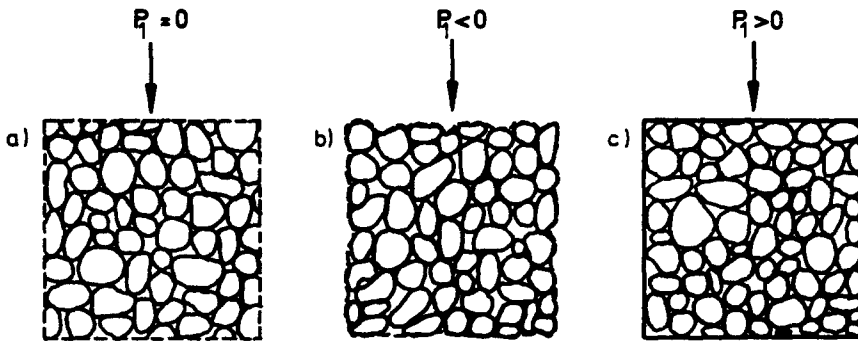


Fig. 1. Intersecting techniques. (a) Statistical cut; (b) grain-to-grain cut; (c) closed solid surface.

$$\gamma_a = k(n^F - n^{F'}). \quad (2)$$

In 1929 and 1930, a polemical discussion on the uplift problem took place between Fillunger and Hoffman, but without further clarification of the problem. In contrast to Fillunger (1914), Hoffman's idea was (Hoffman, 1929) to always use that value of $n^{F'}$ which yields the most unfavourable effect for the respective dam construction, thus increasing the rigidity of the building. This argument was strongly rejected by Fillunger (1929, 1930). Further considerations were given to the uplift problem by Terzaghi (1933) and Terzaghi and Rendulic (1934). It should be mentioned, however, that Terzaghi (1925) had already considered the uplift problem in his famous book "Erdbaumechanik auf bodenphysikalischer Grundlage", where he gave the correct formula for the uplift force. Nevertheless, the fundamental idea of Terzaghi (1933) was to substitute the surface porosity $n^{F'}$ in Fillunger's uplift formula (1) by a quantity n_w which he called the effective surface porosity. In Terzaghi and Rendulic (1934), n_w was determined by means of theoretical investigations in strength of materials and by experiments on concrete specimens. The result was that $n_w \approx 1$ and therefore, the full value of the uplift force must be effective. This result was strongly attacked by Fillunger (1934a-d) who concluded that $n_w \approx 1$ would mean nearly exact point contacts between the grains of the concrete aggregates, thus producing an unrealistic magnitude of stresses (singularities). Nevertheless, Terzaghi adhered to his argument and one can say that his result is close to the correct value of the uplift force; his method of proof, however, has to be refused.

In the subsequent period, Heinrich and Desoyer (1955, 1956) elaborated on the uplift problem with two remarkable papers. Raats (1968) later stated with respect to the considerable confusion concerning the different forces acting on the solid phase of porous media: "The work of Heinrich and Desoyer is an outstanding exemption in this respect, but apparently it has been noted by only a few investigators". However, with respect to the different intersecting techniques, Heinrich and Desoyer supported the results of Fillunger (1913) and concluded that in the case of the statistical cut the uplift vanishes and only in the case of the grain-to-grain cut is the full uplift effective; Raats (1968) also adhered to these results.

Concerning the friction force associated with fluid flow through saturated porous media, again, Fillunger (1914) started the scientific investigations. In Fillunger (1929) (the polemic treatise against Hoffman), a vector equation for the friction force γ_R was given with respect to a volume element of the porous medium, viz.:

$$\gamma_R = n^F(k - \text{grad } p). \quad (3)$$

In (3), which is restricted to water flow through porous media, k is a vertical unit force vector (downwards oriented) and $\text{grad } p$ the gradient of the liquid pressure p . In contrast to Fillunger, Hoffman (1929) concluded that

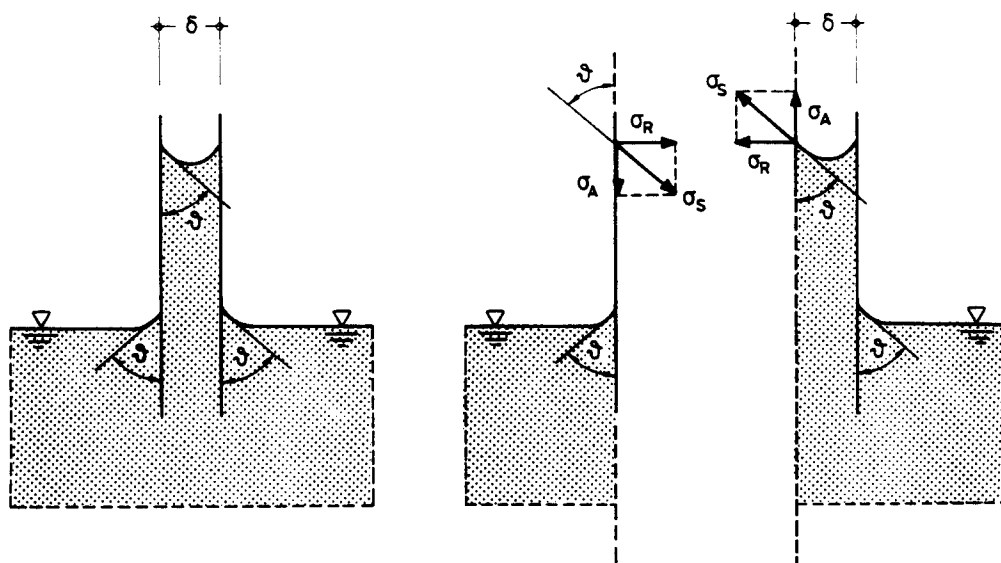


Fig. 2. Profile of liquid interface in circular tubes.

$$\gamma_R = \gamma_w - \text{grad } p \quad (4)$$

where γ_w reduces to k if the pore liquid consists of pure water. Hoffman's result proved to be the correct formula for the friction force. Terzaghi and Rendulic (1934) obtained an equation similar to (3) apart from the fact that n^f was substituted by n_w with $n_w \approx 1$.

Corresponding to Fillunger's results, Heinrich and Desoyer (1956) found a dependence between the friction force and the respective intersecting technique (similar to the uplift question) and Raats (1968) again adhered to this procedure which only leads to the correct result in case of a grain-to-grain cut.

The theoretical treatment of capillary forces in saturated porous media was closely connected with the investigations of the uplift problem. Terzaghi (1933) gave the formula

$$p = \gamma m H_1 \quad (5)$$

where p is the hydrostatic tension for the pore liquid caused by capillarity, γ is the specific weight of the liquid, m is an uplift coefficient similar to n_w , mentioned above and H_1 is the capillary rise. Terzaghi concluded with respect to the vaporization process at the downstream face of masonry dams that the suction for the pore liquid causes an additional pressure for the masonry: "... die in der Verdunstungszone wirksame Oberflächenspannung des Wassers erzeugt im Bereich der Luftseite der Mauer eine zusätzliche Druckbeanspruchung, die durch eine gleichgroße Zugspannung im strömenden Porenwasser ausgeglichen wird". This point of view can only be explained if one considers the profile of liquid interface, e.g. in circular tubes, see Fig. 2, where the surface stresses σ_S of the meniscus cause an axial pressure σ_A for the wall of the tube; radial and tangential stresses are neglected. Since Terzaghi found by experiment a value for the capillary rise of 20 m and thus a considerable pressure for the masonry, it was only natural for Fillunger (1934e) to attack Terzaghi's result again which he considered to be completely unrealistic: "Zwar läßt die Natur sich jede Erklärung gefallen, allein sie hält sich nicht daran". In Fillunger (1934e), the different possibilities of (5) were discussed but in the main statement, i.e. that the capillary suction for the fluid causes an additional pressure for the solid phase, Fillunger agreed with Terzaghi. Section 4 will prove that, from a macroscopic point of view, this was a wrong conclusion.

3. LIQUID-SATURATED POROUS SOLIDS: CONTINUUM APPROACH

Proceeding from general porous media theories, the present section concerns a brief review on kinematics, balance laws and the entropy principle for mixtures together with the concept of volume fractions. In addition, the results of the constitutive theory for the model under discussion are offered. Concerning the different notions and several further information, the reader is referred to de Boer and Ehlers (1986) and Ehlers (1989c). In what follows, all functions introduced are assumed to be sufficiently smooth in space-time.

Considering a liquid-saturated porous medium as an immiscible binary mixture of incompressible constituents φ^i with particles X^i , $i = S$ (solid phase), F (liquid phase), a macroscopic formulation implies a model of superimposed continua where at any time t each spatial point \mathbf{x} of the current configuration is simultaneously occupied by particles X^i of constituents φ^i . These particles proceed from different reference positions \mathbf{X}_i . Thus, each constituent is assigned its own motion function, viz. :

$$\mathbf{x} = \chi_i(\mathbf{X}_i, t). \quad (6)$$

The volume fractions

$$n^i = n^i(\mathbf{x}, t) \quad (7)$$

are defined as the local ratios of the constituent volumes V^i with respect to the bulk volume V . Associated with each φ^i is an effective density ρ^{iR} which is defined as the mass of φ^i per unit of V^i and a partial or bulk density ρ^i defined as the mass of φ^i per unit of V . The density functions are related by

$$\rho^i = n^i \rho^{iR}. \quad (8)$$

Constituent incompressibility, as assumed for the present binary model, implies that the effective densities are constant during deformation, i.e. :

$$\rho^{iR} = \text{const.} \quad (9)$$

From (8) and (9), it is evident that constituent incompressibility does not cause macroscopic incompressibility of φ^i since the bulk densities can still change through changes in volume fractions.

Excluding mass exchanges between the solid and liquid phases, the concept of volume fractions and the constituent balance equations are given with respect to de Boer and Ehlers (1986) or Ehlers (1989b,c) as follows.

Concept of volume fractions :

$$n^S + n^F = 1. \quad (10)$$

Balance of mass :

$$\dot{\rho}^i + \rho^i \text{div } \dot{\mathbf{x}}_i = 0. \quad (11)$$

Balance of momentum :

$$\begin{aligned} \text{div } \mathbf{T}^i + \rho^i (\mathbf{b} - \ddot{\mathbf{x}}_i) + \hat{\mathbf{p}}^i &= \mathbf{0}, \\ \hat{\mathbf{p}}^S + \hat{\mathbf{p}}^F &= \mathbf{0}. \end{aligned} \quad (12)$$

Balance of moment of momentum :

$$\begin{aligned}\mathbf{T}^i &= \mathbf{T}^T - \hat{\mathbf{M}}^i, \\ \hat{\mathbf{M}}^S + \hat{\mathbf{M}}^F &= \mathbf{0}.\end{aligned}\quad (13)$$

Balance of energy :

$$\begin{aligned}\rho^i \dot{\hat{e}}^i &= -\hat{\mathbf{p}}^i \cdot \dot{\mathbf{x}}_i + \mathbf{T}^i \cdot \mathbf{L}_i + \rho^i r^i - \text{div } \mathbf{q}^i + \hat{e}^i, \\ \hat{e}^S + \hat{e}^F &= 0.\end{aligned}\quad (14)$$

In these equations, $\hat{\mathbf{p}}^i$, $\hat{\mathbf{M}}^i$ and \hat{e}^i are the supply terms of momentum, moment of momentum and energy representing the transfers to φ^i caused by the other constituent that occupies \mathbf{x} at time t . For the present binary model, the momentum supplies may also be interpreted as interaction forces per unit of bulk volume. The quantities \mathbf{T}^i , e^i , r^i and \mathbf{q}^i are the partial Cauchy stresses, internal energy densities, external heat supplies and heat influx vectors of φ^i . In using the external body force density \mathbf{b} instead of \mathbf{b}^i , it is understood that

$$\mathbf{b} = \mathbf{b}^S = \mathbf{b}^F. \quad (15)$$

The material time derivatives $(\dots)'_i$ are defined by

$$(\dots)'_i = \frac{\partial(\dots)}{\partial t} + \text{grad } (\dots) \cdot \dot{\mathbf{x}}_i, \quad (16)$$

$\dot{\mathbf{x}}_i$ characterizing the constituent velocities of φ^i . Additionally, $\ddot{\mathbf{x}}_i$ are the corresponding constituent accelerations. The symbol grad denotes partial differentiation with respect to the spatial position \mathbf{x} ; div is the divergence operator corresponding to grad . Finally,

$$\mathbf{L}_i = \text{grad } \dot{\mathbf{x}}_i \quad (17)$$

is the spatial velocity gradient of φ^i .

Considering incompressible constituents as used in the present context, (8) and (11) combine to

$$\dot{n}_i^i + n^i \text{div } \dot{\mathbf{x}}_i = 0, \quad (18)$$

i.e. the balance of mass equations reduce to balance equations for the volume fractions. It is well known that incompressibility of all the constituents of the respective medium gives rise to a certain constraint to be incorporated into the entropy inequality of the model. Such a constraint was first suggested by Mills (1966) in the frame of a simple mixture of incompressible Newtonian fluids and later by Craine (1971) and Atkin and Craine (1976b). In the frame of porous media theories, an incompressibility constraint of the same type was first used by Bowen (1980, 1984) and later by de Boer and Ehlers (1986) and Ehlers (1988, 1989b,c). From (18) and the material time derivative of (10) following the motion of one of the two constituents, the constraint for the present model reads

$$n^S \lambda \text{div } \dot{\mathbf{x}}_S + n^F \lambda \text{div } \dot{\mathbf{x}}_F + \lambda \text{grad } n^F \cdot (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) = 0 \quad (19)$$

where λ is a Lagrangian multiplier.

Using (19), the entropy inequality for the present binary model governed by a single absolute temperature function

$$\Theta = \Theta^S = \Theta^F \quad (20)$$

turns out to yield (Ehlers, 1989b,c):

$$\begin{aligned}
& -\dot{\Psi}_S^S - \dot{\Psi}_F^F - \dot{\Theta}H - (\hat{\mathbf{p}}^F - \lambda \text{grad } n^F) \cdot (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) - (\rho^S \mathbf{K}^{TS} - n^S \lambda \mathbf{I}) \cdot \mathbf{L}_S - (\rho^F \mathbf{K}^{TF} - n^F \lambda \mathbf{I}) \cdot \mathbf{L}_F \\
& - \frac{1}{\Theta} \mathbf{h} \cdot \text{grad } \Theta \geq 0 \quad (21)
\end{aligned}$$

where

$$H = H^S + H^F. \quad (22)$$

In these relations Ψ^i and H^i are the constituent free energy and entropy functions per unit of bulk volume,

$$\mathbf{K}^i = \frac{1}{\rho^i} (\Psi^i \mathbf{I} - \mathbf{T}^{Ti}) \quad (23)$$

are the chemical potential tensors as introduced by Bowen (1967) or Bowen and Wiese (1969), respectively, and \mathbf{h} is the influx vector for the respective medium defined by

$$\begin{aligned}
\mathbf{h} &= \mathbf{h}^S + \mathbf{h}^F, \\
\mathbf{h}^i &= \mathbf{q}^i + \Theta H^i \mathbf{u}_i.
\end{aligned} \quad (24)$$

The diffusion velocities

$$\mathbf{u}_i = \dot{\mathbf{x}}_i - \dot{\mathbf{x}} \quad (25)$$

relate the constituent velocities to the mean velocity field

$$\dot{\mathbf{x}} = \frac{1}{\rho} (\rho^S \dot{\mathbf{x}}_S + \rho^F \dot{\mathbf{x}}_F), \quad \rho = \rho^S + \rho^F. \quad (26)$$

Thus, the symbol $(\dots)^\cdot$ characterizes the material time derivative following the mean velocity $\dot{\mathbf{x}}$, i.e.:

$$(\dots)^\cdot = \frac{\partial(\dots)}{\partial t} + \text{grad}(\dots) \cdot \dot{\mathbf{x}}. \quad (27)$$

From the preceding considerations, the present model is defined by the balance equations for the volume fractions (18), the mixture balance equations (12)–(14) and the entropy inequality (21) together with the following set of constitutive postulates which, from the principle of equipresence, must be assumed in the first place as functions of a common set of independent external variables Φ , viz.:

$$(\Psi^i, H, \rho^i \mathbf{K}^{Ti} - n^i \lambda \mathbf{I}, \mathbf{h}, \hat{\mathbf{p}}^F - \lambda \text{grad } n^F) = \mathcal{R}(\Phi), \quad \Phi = \{\Theta, \text{grad } \Theta, \mathbf{F}_S, \overset{\lambda}{\mathbf{G}}_S, \dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S, \mathbf{D}_F\}. \quad (28)$$

In (28)₂,

$$\begin{aligned}
\mathbf{F}_S &= \text{Grad}_S \mathbf{x}, \\
\overset{\lambda}{\mathbf{G}}_S &= \text{Grad}_S \mathbf{F}_S
\end{aligned} \quad (29)$$

are the first and second solid deformation gradients and \mathbf{D}_F is the symmetric part of \mathbf{L}_F . The symbol Grad_i denotes partial differentiation with respect to the reference position of φ^i .

The set Φ defines a homogeneous second-grade medium, cf. Wang (1973), where

$$\begin{aligned}\text{Grad}_i n_{oi}^i &= \mathbf{o}, \\ \text{Grad}_i \rho_{oi}^{iR} &= \mathbf{o},\end{aligned}\quad (30)$$

n_{oi}^i and ρ_{oi}^{iR} characterizing the volume fractions and effective densities in the reference positions of ϕ^i . Note in passing that $(28)_2$ represents the only unique choice of independent variables for the incompressible binary model under discussion to be related from a general set of variables for arbitrary porous media (Ehlers, 1989c). For a modification of $(28)_2$ towards porous media elasto-plasticity in frame of finite theories, the reader is referred to Ehlers (1988, 1989b,c) where a multiplicative decomposition of the first and second solid deformation gradients has been introduced. However, (28) defines an incompressible inviscid elastic skeleton saturated by an incompressible viscous liquid, thus modelling the same type of medium as used, e.g. by Heinrich and Desoyer (1955, 1956). Furthermore, it should be noted that (28) corresponds to Bowen (1984, 5A.4.32) except that \mathbf{D}_F has been added due to fluid viscosity.

For the present model, thermodynamical restrictions result from the dissipation principle (21) together with the constitutive assumptions (28). Using standard arguments (Bowen, 1976) combined with several symmetry and skew-symmetry conditions (de Boer and Ehlers, 1986; Ehlers, 1989c) it is easily seen that a linear expansion about the so-called mixture equilibrium state

$$\Phi \rightarrow \Phi_o = \{\Theta, \text{grad } \Theta = \mathbf{o}, \mathbf{F}_S, \mathring{\mathbf{G}}_S, \dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S = \mathbf{o}, \mathbf{D}_F = \mathbf{O}\} \quad (31)$$

yields the model to be governed by

$$\begin{aligned}\Psi^i, H &= \not\prec(\Theta, \mathbf{F}_S), \\ H &= -\frac{\partial \Psi^i}{\partial \Theta}, \quad \Psi^i = \Psi^S + \Psi^F,\end{aligned}\quad (32)$$

$$\begin{aligned}\rho^S \mathbf{K}^{TS} - n^S \lambda \mathbf{I} &= -\frac{\partial \Psi^i}{\partial \mathbf{F}_S} \mathbf{F}_S^T, \\ \rho^F \mathbf{K}^{TF} - n^F \lambda \mathbf{I} &= -2\mu^F \mathbf{D}_F - \lambda^F (\mathbf{D}_F \cdot \mathbf{I}) \mathbf{I},\end{aligned}\quad (33)$$

$$\mathbf{h} = -\beta_\Theta \text{grad } \Theta + \Theta \alpha_\Theta (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) - \Theta \frac{\partial \Psi^S}{\partial \Theta} \mathbf{u}_S - \Theta \frac{\partial \Psi^F}{\partial \Theta} \mathbf{u}_F, \quad (34)$$

$$\hat{\mathbf{p}}^F - \lambda \text{grad } n^F = -\alpha_\Theta \text{grad } \Theta - \mathbf{S}_v (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) - \mathbf{F}_S^{T-1} \left(\frac{\partial \Psi^F}{\partial \mathbf{F}_S} \mathring{\mathbf{G}}_S \right)^1. \quad (35)$$

In (35), the symbol $(\dots)^1$ defines a contraction of the arguments in brackets towards a vector. In case of isotropic permeability,

$$\mathbf{S}_v = \frac{(n^F)^2 \gamma^{FR}}{k^F} \mathbf{I} \quad (36)$$

where γ^{FR} is the effective specific weight of the pore liquid and k^F the coefficient of permeability.

For an extended derivation of (31)–(36), the reader is referred to Ehlers (1989b,c). In the above equations,

$$\mu^F, \lambda^F, \alpha_\Theta, \beta_\Theta, k^F = \not\prec(\Phi_o) \quad (37)$$

where μ^F and λ^F are the macroscopic shear and bulk viscosities of the liquid phase, α_Θ is

the entropy coupling parameter for the present single temperature model and β_Θ the coefficient of thermal conductivity for the whole system. However, it should be noted that even if the liquid constituent is assumed to be incompressible,

$$\mathbf{D}_F \cdot \mathbf{I} = \text{div } \dot{\mathbf{x}}_F \neq 0, \quad (38)$$

so that λ^F must generally be incorporated into the theory.

Combining (23) and (33), one obtains

$$\begin{aligned} \mathbf{T}^S &= (\Psi^S - n^S \lambda) \mathbf{I} + \frac{\partial \Psi^I}{\partial \mathbf{F}_S} \mathbf{F}_S^T, \\ \mathbf{T}^F &= (\Psi^F - n^F \lambda) \mathbf{I} + 2\mu^F \mathbf{D}_F + \lambda^F (\mathbf{D}_F \cdot \mathbf{I}). \end{aligned} \quad (39)$$

From these equations, it is easily seen that the partial stresses \mathbf{T}^i are symmetric, namely, as a result of a linear expansion of Ψ^i about the state $\Phi = \Phi_0$ (mixture equilibrium).

In case of $\mathbf{D}_F = \mathbf{O}$, the liquid pressure is defined by

$$\mathbf{T}^F = -n^F p \mathbf{I} \quad (40)$$

where

$$p = \lambda - \frac{\Psi^F}{n^F} \quad (41)$$

characterizes the effective pressure of the incompressible pore liquid.

For several purposes it is convenient to use the free energy densities ψ^i instead of the functions Ψ^i .

$$\Psi^i = \rho^i \psi^i. \quad (42)$$

Following this, (35) and (39)–(42) combine to yield

$$\begin{aligned} \mathbf{T}^S &= -n^S p \mathbf{I} + \mathbf{T}_E^S, \\ \mathbf{T}^F &= -n^F p \mathbf{I} + \mathbf{T}_E^F, \\ \hat{\mathbf{p}}^F &= p \text{grad } n^F + \hat{\mathbf{p}}_E^F \end{aligned} \quad (43)$$

where

$$\begin{aligned} \mathbf{T}_E^S &= \left(\rho^S \frac{\partial \psi^S}{\partial \mathbf{F}_S} + \rho^F \frac{\partial \psi^F}{\partial \mathbf{F}_S} \right) \mathbf{F}_S^T, \\ \mathbf{T}_E^F &= 2\mu^F \mathbf{D}_F + \lambda^F (\mathbf{D}_F \cdot \mathbf{I}), \\ \hat{\mathbf{p}}_E^F &= -\alpha_\Theta \text{grad } \Theta - \mathbf{S}_i (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) - \mathbf{F}_S^{T-1} \left(\rho^F \frac{\partial \psi^F}{\partial \mathbf{F}_S} \dot{\mathbf{G}}_S \right)^i. \end{aligned} \quad (44)$$

As is usual in theories of constrained materials, the index $(\dots)_E$ denotes the so-called “extra quantities”; concerning the notion “extra”, the reader is referred to Truesdell and Noll (1965, Section 30). Moreover, when deriving (43) and (44),

$$\frac{\partial \Psi^I}{\partial \mathbf{F}_S} = - \left(\Psi^S - n^S \frac{\Psi^F}{n^F} \right) \mathbf{F}_S^{T-1} + \rho^S \frac{\partial \psi^S}{\partial \mathbf{F}_S} + \rho^F \frac{\partial \psi^F}{\partial \mathbf{F}_S} \quad (45)$$

holds where

$$\begin{aligned} \frac{\partial \rho^S}{\partial \mathbf{F}_S} &= -\rho^S \mathbf{F}_S^{T-1}, \\ \frac{\partial \rho^F}{\partial \mathbf{F}_S} &= \frac{n^S}{n^F} \rho^F \mathbf{F}_S^{T-1} \end{aligned} \quad (46)$$

has been used. The latter equations are a direct consequence of (8)–(10) in connection with

$$n^S = n_{\rho^S}^S (\det \mathbf{F}_S)^{-1} \quad (47)$$

to be obtained by integration of (18). Similarly, it is easily proved that

$$\mathbf{F}_S^{T-1} \left(\frac{\partial \Psi^F}{\partial \mathbf{F}_S} \hat{\mathbf{G}}_S \right)^I = \frac{\Psi^F}{n^F} \text{grad } n^F + \mathbf{F}_S^{T-1} \left(\rho^F \frac{\partial \psi^F}{\partial \mathbf{F}_S} \hat{\mathbf{G}}_S \right)^I. \quad (48)$$

From the above considerations, the medium under discussion is properly defined by the respective balance equations together with the results of the constitutive theory. Finally, note in passing that for the present single temperature model it was not necessary to give constitutive equations for the energy supplies \dot{e}^i since the temperature variation for the whole system must be calculated from the sum of the energy balances (14), viz.:

$$\begin{aligned} -\dot{\Psi}_S^S - \dot{\Psi}_F^F - \Theta \dot{H} - H(\dot{\Theta} + \Theta \text{div } \dot{\mathbf{x}}) - \hat{\mathbf{p}}^F \cdot (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) - \rho^S \mathbf{K}^{TS} \cdot \mathbf{L}_S \\ - \rho^F \mathbf{K}^{TF} \cdot \mathbf{L}_F + \rho r^I - \text{div } \mathbf{h} = 0 \end{aligned} \quad (49)$$

where

$$r^I = \frac{1}{\rho} (\rho^S r^S + \rho^F r^F) \quad (50)$$

denotes the inner part of the external heat supplies as defined by Truesdell and Toupin (1960, Section 243).

Concerning constitutive equations for the free energy densities ψ^i in the elastic range of the solid material, the reader is referred to Ehlers (1989a) where a new finite elasticity law for porous media is introduced. A general expansion of the theory towards elasto-plastic skeletons with respect to finite theories can be taken from Ehlers (1988) and Ehlers (1989b,c). Furthermore, the plastic response of ductile and, especially, granular porous materials will be the subject of a forthcoming report (de Boer and Ehlers, 1989).

4. UPLIFT, FRICTION AND CAPILLARY FORCES IN THE LIGHT OF POROUS MEDIA THEORIES

In using the results of Section 3, the questions of uplift, friction and capillary forces can be clarified on the basis of general porous media theories. In order to compare the following conclusions with the classical results of Section 2, the problem is restricted to the isothermal case. In addition, inertia effects are neglected, thus reducing the general deformation process of the skeleton to the statical case. As is usual in hydraulics, the latter assumption means that liquid flow through porous solids is considered as lingering flow, compare, e.g. Hamel (1934). From the preceding considerations, the momentum balance equations (12) reduce to

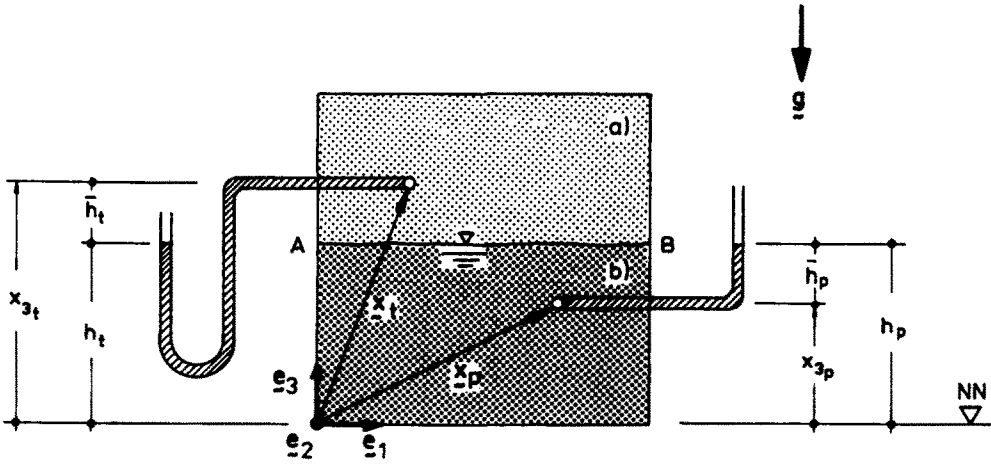


Fig. 3. Liquid-saturated soil with free ground-water level A-B. (a) Capillary zone; (b) flow zone.

$$\begin{aligned}\operatorname{div} \mathbf{T}^S + \rho^S \mathbf{b} - \hat{\mathbf{p}}^F &= \mathbf{0} \\ \operatorname{div} \mathbf{T}^F + \rho^F \mathbf{b} + \hat{\mathbf{p}}^F &= \mathbf{0}\end{aligned}\quad (51)$$

where

$$\mathbf{b} = \mathbf{g}, \quad (52)$$

i.e. the external body force density is identified with the gravity field which can be related from a potential U , viz.:

$$\mathbf{g} = -\operatorname{grad} U. \quad (53)$$

The definitions (51)–(53) together with the assumption of a binary medium of incompressible constituents correspond directly to Heinrich and Desoyer (1955, 1956).

By use of (10), it follows from (43), (51) and (52) that

$$\operatorname{div} \mathbf{T}_E^S - n^S \operatorname{grad} p + \rho^S \mathbf{g} - \hat{\mathbf{p}}_E^F = \mathbf{0} \quad (54)$$

where

$$\begin{aligned}\hat{\mathbf{p}}_E^F &= -\mathbf{z}^F + n^F \operatorname{grad} p - \rho^F \mathbf{g}, \\ \mathbf{z}^F &= \operatorname{div} \mathbf{T}_E^F.\end{aligned}\quad (55)$$

In hydraulics as well as fluid mechanics, \mathbf{z}^F usually defines the viscosity force of the pore liquid. In covering the fields of ground-water flow through soil or pore water flow through earth dams, masonry or concrete dams, respectively, \mathbf{z}^F can generally be neglected in comparison with the other terms incorporated into (55)₁, cf. de Boer and Ehlers (1986, Section 5.1).

In eliminating the extra momentum supply, (54) and (55)₁ combine to yield

$$\operatorname{div} \mathbf{T}_E^S - \operatorname{grad} p + (\rho^S + \rho^F) \mathbf{g} = \mathbf{0}. \quad (56)$$

Further modifications of this equation are due to the pressure head h_p in the flow zone or the suction head h_t in the capillary zone, compare e.g. Fig. 3.

Thus,

$$h = h_p = \frac{p}{\gamma^{FR}} + \frac{U}{|\mathbf{g}|},$$

$$h = h_t = \frac{p}{\gamma^{FR}} + \frac{U}{|\mathbf{g}|}. \quad (57)$$

Since in both cases p defines the effective liquid pressure given by (41), another version of (57)₂ is

$$h = h_t = \frac{U}{|\mathbf{g}|} - \frac{p_t}{\gamma^{FR}} \quad (58)$$

where p_t characterizes the effective liquid suction in the capillary zone.

From (56) and (57), one obtains

$$\operatorname{div} \mathbf{T}_E^S - \gamma^{FR} \operatorname{grad} h + n^S (\gamma^{SR} - \gamma^{FR}) \frac{\mathbf{g}}{|\mathbf{g}|} = \mathbf{0} \quad (59)$$

where (52) and (53) together with the obvious conclusion

$$\operatorname{grad} \gamma^{FR} = \operatorname{grad} (\rho^{FR} |\mathbf{g}|) = \mathbf{0} \quad (60)$$

have been used.

It is easily seen that (59) directly corresponds to Heinrich and Desoyer (1956, eqn (9a)) except for the fact that these authors arrived at their result by use of a special intersecting technique, namely the so-called grain-to-grain cut. The difference between the latter result and (59), however, lies in the fact that the extra solid stresses $\mathbf{T}_E^S \dagger$ are incorporated into (59) which, from (44)₁, are a unique function of the deformation process, whereas concluded from the grain-to-grain cut, Heinrich and Desoyer (1956, eqn (9a)) contains the total partial stress tensor \mathbf{T}^S , compare (43)₁. The reason for this obvious difference is motivated as follows: Heinrich and Desoyer investigated the incompressible model of a liquid-saturated porous solid, thus elaborating on the same type of problem as discussed in Section 3 of the present article. In the frame of a purely mechanical theory, however, Heinrich and Desoyer did not realize an incompressibility constraint of type (19) which made it necessary to use a special intersecting technique to arrive at the desired result. Nevertheless, it seems that Heinrich and Desoyer (1956, p. 83) recognized this fundamental problem, since they wrote: "In einer Kontinuumstheorie können die Deformationen bzw. Deformationsgeschwindigkeiten nur einen Zusammenhang mit den Schnittspannungen besitzen, sie können aber nicht direkt in Zusammenhang gebracht werden mit zusätzlichen Kräften wie z.B. Druckkräften, die von der Flüssigkeit auf das Skelett wirken". The foregoing remarks should not be understood as a criticism of the outstanding work of Heinrich and Desoyer, since in the years 1955 and 1956 on the one hand, mixture theories and porous media theories, respectively, had not been developed, and on the other hand, the method to derive thermodynamical restrictions from the dissipation principle was completely unknown even for single continua and, finally, the handling of incompressible continua, especially in the frame of heterogeneous media, was awaiting research.

In visualizing (59), it is easily seen that this relation consists of three parts which, apart from $\operatorname{div} \mathbf{T}_E^S$, are characterized in the flow zone by the friction force and the specific weight reduced by uplift. Following this, the uplift force is given by

† In soil mechanics, the notion "extra stress" is usually substituted, following Terzaghi, by the notion "effective stress".

$$\mathbf{k}_u = -n^S \gamma^{FR} \frac{\mathbf{g}}{|\mathbf{g}|}, \quad (61)$$

or respectively, by use of (10):

$$\mathbf{k}_u = (n^F - 1) \gamma^{FR} \frac{\mathbf{g}}{|\mathbf{g}|}. \quad (62)$$

On the other hand, if one assumes $\text{grad } h = \mathbf{o}$, which represents the simplest case to be considered, one directly concludes from (53), (57) and (62) that

$$\begin{aligned} \mathbf{k}_u &= (n^F - 1) \text{grad } p, \\ \text{grad } p &= \gamma^{FR} \frac{\mathbf{g}}{|\mathbf{g}|} = \text{const.} \end{aligned} \quad (63)$$

Thus, it is evident that in porous media theories, corresponding to the principle of Archimedes, the unreduced uplift force is effective. Furthermore, (63) is equivalent to the approximate solution of Terzaghi and Rendulic (1934) but corresponds to Fillunger, compare (2), only in case of a selected intersecting technique, namely the grain-to-grain cut. In addition, (61) and (62) equal the results of Heinrich and Desoyer (1956), again, only in case of a grain-to-grain cut.

From (59), the friction force in the flow zone yields, with $h = h_p$,

$$\mathbf{k}_f = -\gamma^{FR} \text{grad } h_p, \quad (64)$$

or respectively, by use of (53), (57)₁ and (60):

$$\mathbf{k}_f = \gamma^{FR} \frac{\mathbf{g}}{|\mathbf{g}|} - \text{grad } p. \quad (65)$$

These results directly correspond to Hoffman (1929), to the approximate solution of Terzaghi and Rendulic (1934) and, in case of a grain-to-grain cut, to Fillunger (1929); compare (3) in the case when the pore liquid consists of pure water. Finally, it corresponds to Heinrich and Desoyer (1956) as well.

Finally, since (59) is not only valid in the flow zone but also in the capillary zone, one easily concludes the capillary force to yield

$$\begin{aligned} \mathbf{k}_c &= -\gamma^{FR} \text{grad } h_c \\ &= \gamma^{FR} \frac{\mathbf{g}}{|\mathbf{g}|} + \text{grad } p, \end{aligned} \quad (66)$$

where (58) has been used.

The effective liquid suction is given by

$$\begin{aligned} p_i &= -p = \gamma^{FR} \left(\frac{U}{|\mathbf{g}|} - h_i \right) \\ &= \gamma^{FR} \bar{h}_i, \end{aligned} \quad (67)$$

\bar{h}_i , characterizing the capillary rise, compare, e.g. Fig. 3. Thus, (67) directly corresponds to the approximate solution of Terzaghi (1933) when using $m = n_w \approx 1$, compare (5).

Concerning the solid phase (56) and (67) combine to

$$\operatorname{div} (\mathbf{T}_E^S + p_t \mathbf{I}) + (\rho^S + \rho^F) \mathbf{b} = \mathbf{0}. \quad (68)$$

From (68), it is easily seen that, in frame of a macroscopic theory, the stress

$$\mathbf{T}^S = \mathbf{T}_E^S + p_t \mathbf{I} \quad (69)$$

in the capillary zone is effected by an additional hydrostatic tension p_t , and not, as assumed by Terzaghi (1933) or Fillunger (1934e), by an additional capillary pressure.

Finally, it should be noted that friction and capillary forces only occur if $\operatorname{grad} h \neq \mathbf{0}$, i.e. in the case of flow processes in the flow zone or in the case of vaporization processes at the downstream face of the suction zone, respectively. The uplift force, however, is always effective.

5. CONCLUDING REMARKS

In the present paper, the question of uplift, friction and capillary forces has been discussed on the basis of porous media theories, thus using a unified macroscopic approach to liquid-saturated porous media. In frame of this procedure, the concept of volume fractions according to the Delesseian law implies that the statistical cut is the only acceptable intersecting technique. Other intersecting techniques as introduced by Fillunger (the grain-to-grain cut or the closed solid surface) can only be used as a boundary condition for the respective surface of the medium and not in the frame of an arbitrary internal cut.

In using the same type of model as Heinrich and Desoyer, i.e. an incompressible skeleton saturated by an incompressible liquid, it could be shown via thermodynamics of heterogeneous media (including an incompressibility constraint to be incorporated into the dissipation principle of the model) that the unreduced values of the uplift, friction and capillary forces must be applied. This obvious result corresponds to the results of Fillunger or Heinrich and Desoyer, respectively, only by use of an internal grain-to-grain intersecting technique. The results of Hoffman, Terzaghi and Terzaghi and Rendulic are effectively or approximately correct, apart from the fact that Terzaghi (and Fillunger) concluded the effective liquid suction in the capillary zone to yield a capillary pressure for the skeleton. As mentioned in Section 2, this obvious misunderstanding can only be explained in connection with the profile of liquid interface, e.g. in circular tubes, compare Fig. 2, when only the axial pressure for the wall of the tube is considered and not the radial or tangential stresses as well. Regarding this artificial approach as belonging more to a microscopic than a macroscopic point of view, we end the article with a quotation from Fillunger (1934e, p. 12): "Die voranstehenden Betrachtungen sind nicht vollkommen befriedigend, da von den drei Hauptnormalspannungen des Spannungszustands, der als Folge der Oberflächenspannungen des Wassers in einem durchtränkten porösen Körper entsteht, nur eine, nämlich der Kapillardruck σ berechnet wurde. In einem Rohr herrschen außer dem Achsialdruck σ noch tangentielle und radiale Spannungen. Diesen entsprechen im durchtränkten porösen Körper ebenfalls gewisse Spannungen, doch kann über sie nur wenig Verlässliches ausgesagt werden".

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